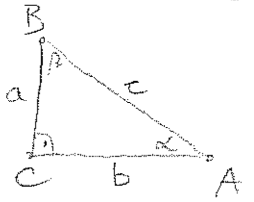
PRIMJENA TRIGONOMETRIJE U PLANIMETRIJI

Osnovne definicije i relacije među trigonometrijskim funkcijama

PRAVOKUTNI TROKUT



$\sin\alpha = \frac{a}{c}$ $\quad$ $\sin\beta = \frac{b}{c}$

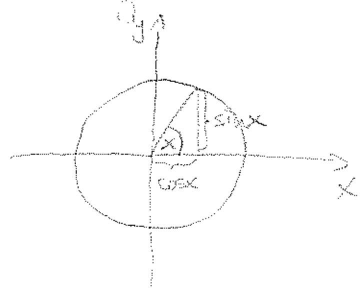
$\cos\alpha = \frac{b}{c}$ $\quad$ $\cos\beta = \frac{a}{c}$

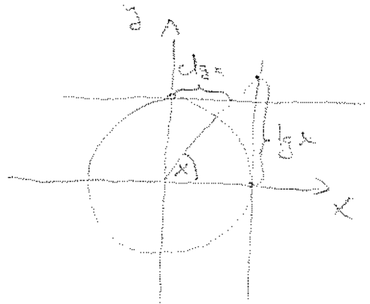
$\text{tg}\,\alpha = \frac{a}{b}$ $\quad$ $\text{tg}\,\beta = \frac{b}{a}$

$\text{ctg}\,\alpha = \frac{b}{a}$ $\quad$ $\text{ctg}\,\beta = \frac{a}{b}$

$\sin 0^\circ = 0$	$\sin 90^\circ = 1$	
$\cos 0^\circ = 1$	$\cos 90^\circ = 0$	
$\text{tg}\,0^\circ = 0$	$\text{tg}\,90^\circ$ ×	
$\text{ctg}\,0^\circ$ ×	$\text{ctg}\,90^\circ = 0$	
$\sin 30^\circ = \frac{1}{2}$	$\sin 45^\circ = \frac{\sqrt{2}}{2}$	$\sin 60^\circ = \frac{\sqrt{3}}{2}$
$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\cos 45^\circ = \frac{\sqrt{2}}{2}$	$\cos 60^\circ = \frac{1}{2}$
$\text{tg}\,30^\circ = \frac{\sqrt{3}}{3}$	$\text{tg}\,45^\circ = 1$	$\text{tg}\,60^\circ = \sqrt{3}$
$\text{ctg}\,30^\circ = \sqrt{3}$	$\text{ctg}\,45^\circ = 1$	$\text{ctg}\,60^\circ = \frac{\sqrt{3}}{3}$

BROJEVNA KRUŽNICA





$-1 \le \sin x \le 1 \qquad -1 \le \cos x \le 1$

$r = 1$ $\qquad$ $r = 1$

$\sin(x + 360^\circ) = \sin x$ $\quad$ $\text{tg}(x + 180^\circ) = \text{tg}\,x$

$\cos(x + 360^\circ) = \cos x$ $\quad$ $\text{ctg}(x + 180^\circ) = \text{ctg}\,x$ $\quad$] periodičnost

$\cos(-x) = \cos x$ $\quad$ $\text{tg}(-x) = -\text{tg}\,x$

$\sin(-x) = -\sin x$ $\quad$ $\text{ctg}(-x) = -\text{ctg}\,x$ $\quad$] parnost/neparnost

- $\text{tg}\,x = \frac{\sin x}{\cos x}$ $\quad$ $\text{ctg}\,x = \frac{\cos x}{\sin x}$ $\quad$ $\text{tg}\,x \cdot \text{ctg}\,x = 1$ $\quad$ $\sin^2 x + \cos^2 x = 1$
- $\sin(90^\circ - x) = \cos x$ $\quad$ $\cos(90^\circ - x) = \sin x$ $\quad$ $\text{tg}(90^\circ - x) = \text{ctg}\,x$
- $\sin(180^\circ - x) = \sin x$ $\quad$ $\cos(180^\circ - x) = -\cos x$ $\quad$ $\text{tg}(180^\circ - x) = -\text{tg}\,x$
- ADICIJSKE FORMULE: $\sin(x+y) = \sin x \cos y + \cos x \sin y$

  $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- FORMULE PRETVORBE SUME U PRODUKT:

  $\sin x + \sin y = 2 \sin\frac{x+y}{2} \cos\frac{x-y}{2}$ $\quad$ $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

  $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ $\quad$ $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$
- FORMULE DVOSTRUKOG KUTA: $\sin 2x = 2\sin x \cos x$

  $\cos 2x = \cos^2 x - \sin^2 x$
- FORMULE POLOVIČNOG KUTA: $\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$ $\quad$ $\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$

- $t = \operatorname{tg}\frac{x}{2}$ $\quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad \operatorname{tg} x = \frac{2t}{1-t^2}$
- Ako je $\alpha+\beta+\gamma = 180^\circ$ (npr. kutovi trokuta) vrijede formule:

$$\cos\alpha + \cos\beta + \cos\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1$$

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}\gamma$$

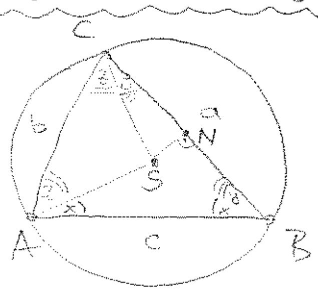
Sve se slično pokazuju pa npr. prva ide ovako:

$$\cos\alpha + \cos\beta + \cos\gamma = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \cos(\alpha+\beta) = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - 2\cos^2\frac{\alpha+\beta}{2} + 1 =$$

$$= 2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right) + 1 = 2\cos\frac{\alpha+\beta}{2}\cdot(-2)\sin\frac{\alpha}{2}\cdot\left(-\sin\frac{\beta}{2}\right) + 1$$

$$= 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1$$

TEOREM O SINUSU



$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R}$$

D: $\sphericalangle BSC = 2\alpha \Rightarrow \sphericalangle BSN = \alpha$

$\triangle BSN: \sin\alpha = \frac{\frac{a}{2}}{R} = \frac{a}{2R} \Rightarrow \frac{a}{\sin\alpha} = 2R$

Slično za $\frac{b}{\sin\beta}, \frac{c}{\sin\gamma}$.

FORMULE ZA POVRŠINU

trokut

$$\boxed{P = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta} = \frac{abc}{4R}$$

četverokut

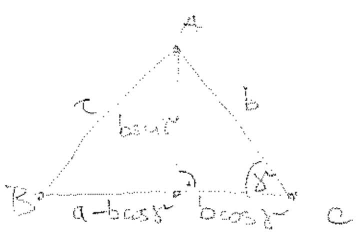
$$\boxed{P = \frac{1}{2}ef\sin\varphi}$$

e, f - dijagonale

TEOREM O KOSINUSU

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}}$$

D:



$$c^2 = (a-b\cos\gamma)^2 + (b\sin\gamma)^2$$

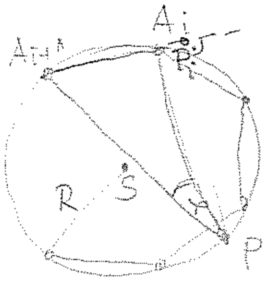
$$= a^2 + b^2\cos^2\gamma - 2ab\cos\gamma + b^2\sin^2\gamma$$

$$= a^2 + b^2 - 2ab\cos\gamma$$

Trigonometrija je vrlo korisna u dokazivanju nekih geometrijskih identiteta i nejednakosti, ali vidjet ćemo i još neke tipove zadataka u kojima trigonometrijska tehnika dobro dođe.

Zad. 1. Točka P nalazi se na kružnici opisanoj oko pravilnog mnogokuta $A_1A_2\ldots A_n$. Ortogonalne projekcije točke P na pravce na kojima leže njegove stranice označene su s $P_1, P_2, \ldots, P_n$. Dokaži da produkt $\prod_{i=1}^{n} \frac{|PA_i|^2}{|PP_i|}$ ne ovisi o izboru točke P.

Rj.



U danom produktu se pojavljuje n članova oblika $\frac{|PA_i|\cdot|PA_{i+1}|}{|PP_i|}$, $i=1,\ldots,n$ $(A_{n+1}=A_1)$.

Promotrimo $\Delta A_iA_{i+1}P$, $i=1,\ldots,n$, uz identifikaciju $A_{n+1}=A_1$

$$P(\Delta A_iA_{i+1}P)=\frac{1}{2}|PA_i|\cdot|PA_{i+1}|\sin\varphi=\frac{1}{2}|A_iA_{i+1}|\cdot|PP_i| \quad \left(\text{metoda površina}\right)$$

$$\frac{|PA_i|\cdot|PA_{i+1}|}{|PP_i|}=\frac{|A_iA_{i+1}|}{\sin\varphi}=2R \quad (\text{sinusov poučak})$$

Množenjem ovih n jednakosti slijedi:

$$\prod_{i=1}^{n}\frac{|PA_i|^2}{|PP_i|}=2^n\cdot R^n=\text{const.}$$

Zad. 2. Dokaži: ako su a, b, c, d redom duljine susjednih stranica konveksnog četverokuta, α kut između stranica a i b, a γ kut između stranica c i d, onda je površina tog četverokuta dana sa

$$P=\sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\frac{\alpha+\gamma}{2}},$$

gdje je $2s=a+b+c+d$ opseg četverokuta (Heronova formula za četverokut).

Rj.

$$2P=ab\sin\alpha+cd\sin\gamma \,/^2$$

$$4P^2=a^2b^2\sin^2\alpha+c^2d^2\sin^2\gamma+2abcd\sin\alpha\sin\gamma$$

$$4P^2=a^2b^2+c^2d^2+2abcd\sin\alpha\sin\gamma-(a^2b^2\cos^2\alpha+c^2d^2\cos^2\gamma) \qquad (1)$$

Kosinusov poučak (dvostruka primjena) daje:

$$a^2+b^2-2ab\cos\alpha=c^2+d^2-2cd\cos\gamma$$

$$a^2+b^2-c^2-d^2=2ab\cos\alpha-2cd\cos\gamma \,/^2$$

$$(a^2+b^2-c^2-d^2)^2=4(a^2b^2\cos^2\alpha+c^2d^2\cos^2\gamma-2abcd\cos\alpha\cos\gamma)$$

Iz adicijskih formula imamo:

$$\cos(\alpha+\gamma)=\cos\alpha\cos\gamma-\sin\alpha\sin\gamma$$

Ta dva rezultata uvrstimo u (1) i dobivamo:

$$4P^2=a^2b^2+c^2d^2-\frac{1}{4}(a^2+b^2-c^2-d^2)^2-2abcd\cos(\alpha+\gamma)\Big) /\cdot 4$$

$$16P^2=4a^2b^2+4c^2d^2-(a^2+b^2-c^2-d^2)^2-8abcd\cos(\alpha+\gamma)$$

$$16P^2=4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2-8abcd\,(1+\cos(\alpha+\gamma))=$$

$$=\left[2(ab+cd)-a^2-b^2+c^2+d^2\right]\cdot\left[2(ab+cd)+a^2+b^2-c^2-d^2\right]-16abcd\cos^2\frac{\alpha+\gamma}{2}$$

$$=\left[(c+d)^2-(a-b)^2\right]\cdot\left[(a+b)^2-(c-d)^2\right]-16abcd\cos^2\frac{\alpha+\gamma}{2}$$

$$=(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)-16abcd\cos^2\frac{\alpha+\gamma}{2}$$

$$=2(s-a)\cdot 2(s-b)\cdot 2(s-c)\cdot 2(s-d)-16abcd\cos^2\frac{\alpha+\gamma}{2}$$

$$=16(s-a)(s-b)(s-c)(s-d)-16abcd\cos^2\frac{\alpha+\gamma}{2}$$

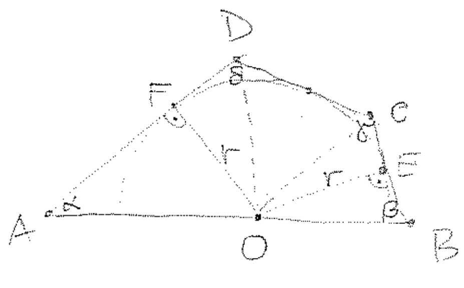
$$\Rightarrow P=\sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\frac{\alpha+\gamma}{2}}$$

Nap. Za tetivni četverokut je $\alpha+\gamma=180^\circ \Rightarrow P=\sqrt{(s-a)(s-b)(s-c)(s-d)}$, Brahmaguptina formula za površinu tetivnog četverokuta.

Zad.3. Kružnica ima središte na stranici $\overline{AB}$ konveksnog tetivnog četverokuta $ABCD$. Preostale tri stranice dodiruju tu kružnicu. Dokaži da je

$$|AD|+|BC|=|AB|.$$

Rj.



$$\mathrm{ctg}\,\alpha=\frac{|AF|}{r}\Rightarrow |AF|=r\,\mathrm{ctg}\,\alpha$$

$$\mathrm{ctg}\,\beta=\frac{|BE|}{r}\Rightarrow |BE|=r\,\mathrm{ctg}\,\beta$$

$$\mathrm{ctg}\,\frac{\gamma}{2}=\frac{|CE|}{r}\Rightarrow |CE|=r\,\mathrm{ctg}\,\frac{\gamma}{2}$$

$$\mathrm{ctg}\,\frac{\delta}{2}=\frac{|DF|}{r}\Rightarrow |DF|=r\,\mathrm{ctg}\,\frac{\delta}{2}$$

$$\sin\alpha=\frac{r}{|AO|}\Rightarrow |AO|=\frac{r}{\sin\alpha}$$

$$\sin\beta=\frac{r}{|BO|}\Rightarrow |BO|=\frac{r}{\sin\beta}$$

$$|AB|=|AO|+|BO|=r\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}\right)$$

$$|AD|+|BC|=|AF|+|DF|+|BE|+|CE|=$$

$$=r\Big(\mathrm{ctg}\,\alpha+\mathrm{ctg}\,\beta+\underbrace{\mathrm{ctg}\,\tfrac{\gamma}{2}}_{=\mathrm{tg}\,\frac{\alpha}{2}}+\underbrace{\mathrm{ctg}\,\tfrac{\delta}{2}}_{=\mathrm{tg}\,\frac{\beta}{2}}\Big)$$

(zbog tetivnosti od $ABCD$ $\alpha+\gamma=\beta+\delta=180^\circ$)

$$\mathrm{ctg}\,\alpha+\mathrm{tg}\,\frac{\alpha}{2}+\mathrm{ctg}\,\beta+\mathrm{tg}\,\frac{\beta}{2}=$$

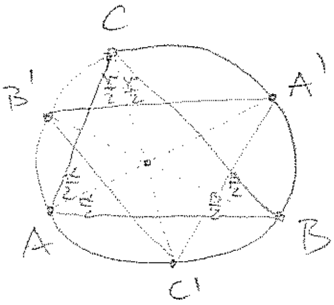
$$=\frac{1}{\mathrm{tg}\,\alpha}+\mathrm{tg}\,\frac{\alpha}{2}+\frac{1}{\mathrm{tg}\,\beta}+\mathrm{tg}\,\frac{\beta}{2}=\frac{1-\mathrm{tg}^2\frac{\alpha}{2}}{2\,\mathrm{tg}\frac{\alpha}{2}}+\mathrm{tg}\,\frac{\alpha}{2}+\frac{1-\mathrm{tg}^2\frac{\beta}{2}}{2\,\mathrm{tg}\frac{\beta}{2}}+\mathrm{tg}\,\frac{\beta}{2}=$$

$$=\frac{1+\mathrm{tg}^2\frac{\alpha}{2}}{2\,\mathrm{tg}\frac{\alpha}{2}}+\frac{1+\mathrm{tg}^2\frac{\beta}{2}}{2\,\mathrm{tg}\frac{\beta}{2}}=\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}\ \checkmark$$

Zad. 4. Polumjer opisane kružnice k oko trokuta ABC je R. Simetrale kutova trokuta ABC ponovno sijeku kružnicu u točkama A', B', C'. Ako je P površina trokuta ABC, a Q površina trokuta A'B'C', dokaži da je

$$16Q^3 \geq 27R^4 \cdot P.$$

Rj.



α, β, γ – kutovi ΔABC

α', β', γ' – kutovi $\Delta A'B'C'$

$$\alpha' = \sphericalangle B'A'C' = \sphericalangle B'A'A + \sphericalangle AA'C' = \sphericalangle B'BA + \sphericalangle ACC'$$
$$= \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\beta+\gamma}{2}$$

Analogno: $\beta' = \frac{\alpha+\gamma}{2}$, $\gamma' = \frac{\alpha+\beta}{2}$

$$\sin\alpha' = \sin\frac{\beta+\gamma}{2} = \sin\left(90^\circ - \frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}$$

Analogno: $\sin\beta' = \cos\frac{\beta}{2}$, $\sin\gamma' = \cos\frac{\gamma}{2}$

$$P = \frac{abc}{4R} = 2R^2 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = 2R^2 \sin\alpha \sin\beta \sin\gamma$$

Analogno: $Q = \frac{a'b'c'}{4R} = 2R^2 \sin\alpha' \sin\beta' \sin\gamma' = 2R^2 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

A-G nejednakost daje: $\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \geq \sqrt[3]{\sin\alpha \sin\beta \sin\gamma}$

$$\left(4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)^3 \geq 27 \sin\alpha \sin\beta \sin\gamma$$

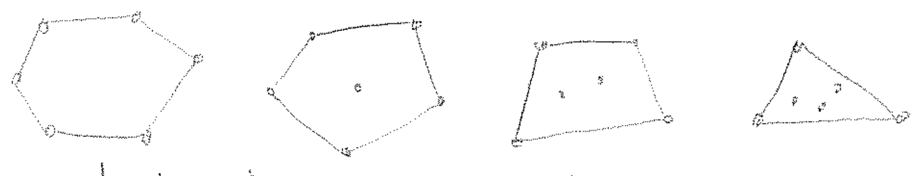
$$\left(4 \cdot \frac{Q}{2R^2}\right)^3 \geq 27 \cdot \frac{P}{2R^2}$$

$${}^{16}64 \cdot \frac{Q^3}{2^3R^6} \geq 27 \cdot \frac{P}{2R^2} \Big/ \cdot 2R^6$$

$$16Q^3 \geq 27R^4 \cdot P$$

Zad. 5. Neka je d najkraća udaljenost, a D najdulja udaljenost između šest zadanih točaka u ravnini. Dokaži da je uvijek $\frac{D}{d} \geq \sqrt{3}$.

Rj. Promotrimo sve međusobne položaje između šest točaka u ravnini. One su ili sve kolinearne ili su u jednom od sljedećih položaja:



(Točke iz unutrašnjosti mogu ležati i na stranicama.)

Zaključujemo da uvijek (osim u slučaju da su sve kolinearne ili da ih više leži na jednom pravcu kada je nejednakost očita) možemo pronaći ΔABC u kojemu je kut pri jednom vrhu (npr. A) veći ili jednak $120°$. Iz kosinusovog poučka tada slijedi:

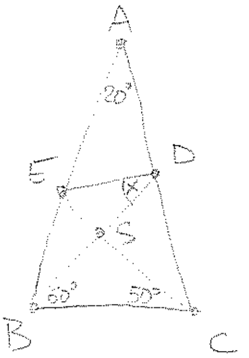
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \geq b^2 + c^2 - 2bc\underbrace{\cos 120°}_{=-\frac{1}{2}} = b^2 + c^2 + bc$$

Budući vrijedi $d \leq a, b, c \leq D$ imamo:

$$D^2 \geq a^2 \geq b^2 + c^2 + bc \geq d^2 + d^2 + d^2 = 3d^2 \Rightarrow \frac{D}{d} \geq \sqrt{3}.$$

<u>Zad. 6.</u> Zadan je jednakokračni trokut ABC kod kojeg je $|AB| = |AC|$ i $\sphericalangle BAC = 20°$. Neka su točke D i E na stranicama $\overline{AC}$ i $\overline{AB}$ redom uzete tako da je $\sphericalangle ECB = 50°$ i $\sphericalangle DBC = 60°$. Odredi kut $\sphericalangle EDB$.

<u>Rj.</u>



$\sphericalangle EDB = x$

ΔABC je jednakokračan ($|AB| = |AC|$) $\Rightarrow \sphericalangle ABC = \sphericalangle BCA$

$\sphericalangle ABC = \frac{1}{2}(180° - \sphericalangle BAC) = \frac{1}{2}(180° - 20°) = 80°$

$\sphericalangle ABD = \sphericalangle ABC - \sphericalangle DBC = 80° - 60° = 20°$

$\sphericalangle ECD = \sphericalangle BCA - \sphericalangle ECB = 80° - 50° = 30°$

Iz ΔBCE je: $\sphericalangle BEC = 180° - \sphericalangle EBC - \sphericalangle ECB = 180° - 80° - 50° = 50°$

$\Rightarrow \sphericalangle BEC = \sphericalangle ECB \Rightarrow \Delta BCE$ je jednakokračan i $|BE| = |BC|$

Po sinusovom poučku (ΔBDE): $\frac{|BD|}{|BE|} = \frac{\sin\sphericalangle BED}{\sin\sphericalangle EDB} = \frac{\sin(160° - x)}{\sin x}$

$\sphericalangle CDB = 180° - \sphericalangle CBD - \sphericalangle BCD = 180° - 60° - 80° = 40°$

Po sinusovom poučku (ΔBCD): $\frac{|BD|}{|BC|} = \frac{\sin\sphericalangle BCD}{\sin\sphericalangle CDB} = \frac{\sin 80°}{\sin 40°} = \frac{2\sin 40°\cos 40°}{\sin 40°} = 2\cos 40°$

Zbog $|BC| = |BE|$ slijedi $\frac{\sin(160° - x)}{\sin x} = 2\cos 40°$

$\sin(160° - x) = \sin(180° - (160° - x)) = \sin(x + 20°) = \sin x\cos 20° + \sin 20°\cos x$

$2\cos 40°\sin x = 2\cos(60° - 20°)\sin x = 2(\cos 60°\cos 20° + \sin 60°\sin 20°)\sin x =$

$= 2\left(\frac{1}{2}\cos 20° + \frac{\sqrt{3}}{2}\sin 20°\right)\sin x = \cos 20°\sin x + \sqrt{3}\sin 20°\sin x$

Izjednačavanjem dobivamo:

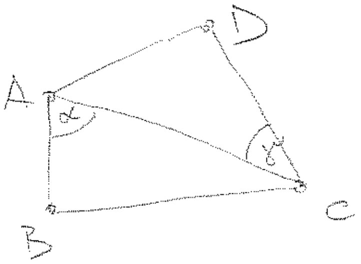
$$\cancel{\sin 20°}\cos x = \sqrt{3}\,\cancel{\sin 20°}\sin x$$

$$\operatorname{tg} x = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \boxed{x = 30°}$$

Zad. 7. Površina konveksnog četverokuta u ravnini je $32\,cm^2$, a suma duljina njegovih nasuprotnih stranica i jedne dijagonale jednaka je 16 cm. Odredi sve vrijednosti koje može imati duljina druge dijagonale.

Rj:

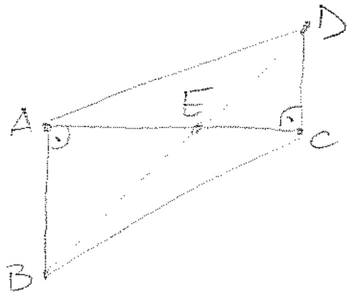


Neka je u četverokutu $ABCD$
$|AB|+|AC|+|CD|=16$.

Vrijedi $2P=|AB|\cdot|AC|\sin\alpha+|AC|\cdot|CD|\sin\gamma$.
Nadalje, vrijedi $2P\leq|AB|\cdot|AC|+|AC|\cdot|CD|=|AC|(|AB|+|CD|)$, odnosno
$2P\leq|AC|(16-|AC|)$ pri čemu jednakost vrijedi ako i samo ako je $\alpha=\gamma=90^\circ$. Imamo (zbog A-G nejednakosti):

$$2P\leq|AC|(16-|AC|)\leq\left(\frac{|AC|+16-|AC|}{2}\right)^2\leq\frac{256}{4}=64 \Rightarrow P\leq 32$$

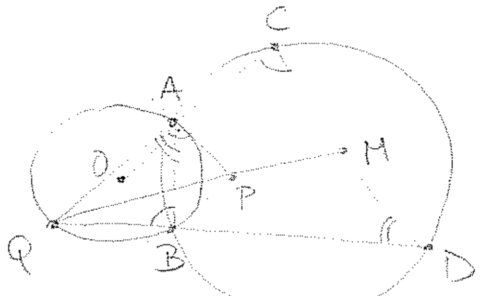
Budući znamo da je $P=32$, u prethodnim nejednakostima zapravo imamo jednakosti pa je stoga $|AC|=8$ i $\alpha=\gamma=90^\circ$.
Sada dobivamo ovakvu sliku:



Imamo i $|AC|=|AB|+|CD|=8$, a to je moguće ako i samo ako su trokuti $\triangle ABE$ i $\triangle DEC$ jednakokračni pravokutni. Tako dobivamo $|BD|=|BE|+|ED|=|AB|\sqrt{2}+|CD|\sqrt{2}=8\sqrt{2}$ cm i to je jedina moguća vrijednost duljine te druge dijagonale.

Zad. 8. Kružnice S_1 i S_2 sijeku se u točkama A i B. Na kružnici S_1 izabrana je točka Q. Pravci QA i QB sijeku kružnicu S_2 u točkama C i D, a tangente na S_1 u točkama A i B sijeku se u točki P. Točka Q je izvan S_2, a točke C i D su izvan S_1. Dokažite da pravac QP prolazi polovištem dužine $\overline{CD}$.

Rj.



Označimo sa M točku presjeka pravaca QP i CD.

$\sphericalangle QAB = 180^\circ - \sphericalangle CAB = \sphericalangle BDC$

Analogno: $\sphericalangle QBA = \sphericalangle ACD$

Primjenom sinusovog poučka na $\triangle CMQ$ i $\triangle DMQ$ dobivamo:

$$\frac{|CM|}{|DM|} = \frac{|CM|/|QM|}{|DM|/|QM|} = \frac{\sin\sphericalangle AQP}{\sin\sphericalangle BQP} \cdot \frac{\sin\sphericalangle BDC}{\sin\sphericalangle ACD}$$

Primjenom sinusovog poučka na $\triangle QAP$ i $\triangle QBP$ dobivamo:

$$1 = \frac{|AP|}{|BP|} = \frac{|AP|/|QP|}{|BP|/|QP|} = \frac{\sin\sphericalangle AQP}{\sin\sphericalangle BQP} \cdot \frac{\sin\sphericalangle QBP}{\sin\sphericalangle QAP} \Rightarrow \frac{\sin\sphericalangle QAP}{\sin\sphericalangle QBP} = \frac{\sin\sphericalangle AQP}{\sin\sphericalangle BQP}$$

$\sphericalangle QAP = 90^\circ + \sphericalangle QAO = 90^\circ + \frac{1}{2}(180^\circ - \sphericalangle AOQ) = 180^\circ - \frac{1}{2}\sphericalangle AOQ = 180^\circ - \sphericalangle QBA = 180^\circ - \sphericalangle ACD$

Analogno: $\sphericalangle QBP = 180^\circ - \sphericalangle QAB = 180^\circ - \sphericalangle BDC$

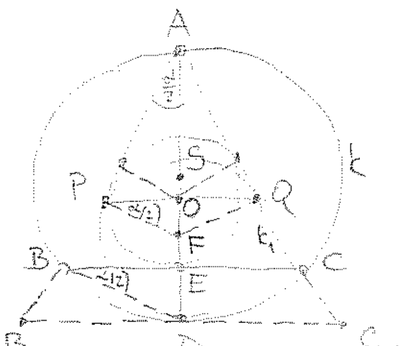
$\Rightarrow \sin\sphericalangle QAP = \sin\sphericalangle ACD$ i $\sin\sphericalangle QBP = \sin\sphericalangle BDC$

$$\Rightarrow \frac{|CM|}{|DM|} = \frac{\sin\sphericalangle QAP}{\sin\sphericalangle QBP} \cdot \frac{\sin\sphericalangle BDC}{\sin\sphericalangle ACD} = 1 \Rightarrow |CM| = |DM|$$

$\Rightarrow M$ je polovište od $\overline{CD}$

Zad. 9. Jednakokračnom trokutu ABC opisana je kružnica k. Kružnica k_1 dodiruje iznutra kružnicu k i krakove trokuta $\overline{AB}$ i $\overline{AC}$ u točkama P i Q. Dokaži da dužina $\overline{PQ}$ sadrži središte trokutu ABC upisane kružnice.

Rj.



Homotetija sa središtem u točki A i koeficijentom $k = \frac{|AE|}{|AD|}$ preslikava $\triangle AB_1C_1$ u $\triangle ABC$. Tom se homotetijom kružnica k_1 upisana trokutu AB_1C_1 preslikava u kružnicu upisanu trokutu ABC, zato je dovoljno pokazati da se tom homotetijom točka F (središte kružnice k_1) preslikava u polovište O dužine $\overline{PQ}$, tj. da je $\frac{|AO|}{|AF|} = k$.

Lako se vidi: $\sphericalangle BAD = \frac{\alpha}{2}$; $\sphericalangle APF = \sphericalangle ABD = 90^\circ$.

$$k = \frac{|AE|}{|AD|} = \frac{|AB|\cos\frac{\alpha}{2}}{\frac{|AB|}{\cos\frac{\alpha}{2}}} = \cos^2\frac{\alpha}{2} \quad \text{(pravokutni } \triangle ABE, \triangle ABD)$$

$$\frac{|AO|}{|AF|} = \frac{|AP|\cos\frac{\alpha}{2}}{\frac{|AP|}{\cos\frac{\alpha}{2}}} = \cos^2\frac{\alpha}{2} = k \quad \text{(pravokutni } \triangle APO, \triangle APF)$$

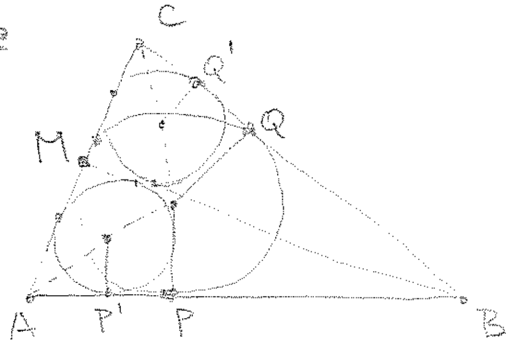
Nap. Prethodni zadatak pokazuje kako su sličnost i trigonometrija ponekad u uskoj vezi, štoviše, ponekad mogu zamijeniti jedna drugu.

Zad. 10. Na stranici $\overline{AC}$ trokuta ABC odabrana je točka M tako da su polumjeri kružnica upisanih trokutima ABM i MBC jednaki. Dokaži da je

$$|BM|^2 = P \cdot \operatorname{ctg}\frac{\beta}{2},$$

gdje je P površina trokuta ABC, $\beta = \sphericalangle ABC$.

Rj.



Oznake za površine, poluopsege i polumjere upisanih kružnica:

$\Delta ABC \rightsquigarrow P, s, r$

$\Delta ABM \rightsquigarrow P_1, s_1, \varrho$

$\Delta BCM \rightsquigarrow P_2, s_2, \varrho$

Imamo: $P = sr$, $P_1 = s_1\varrho$, $P_2 = s_2\varrho$ i odatle: $sr = (s_1+s_2)\varrho$.

Također imamo $s_1+s_2 = s+|BM|$, a onda i $\frac{\varrho}{r} = \frac{s}{s+|BM|}$. (1)

Iz sličnosti trokuta dobivamo omjere:

$$\frac{\varrho}{r} = \frac{|AP'|}{|AP|} \quad i \quad \frac{\varrho}{r} = \frac{|CQ'|}{|CQ|}$$

Pripadni odsječci upisanih kružnica u odgovarajuće trokute računaju se

$|AP| = s - |BC|$ $\qquad$ $|CQ| = s - |AB|$

$|AP'| = s_1 - |BM|$ $\qquad$ $|CQ'| = s_2 - |BM|$

$$\frac{\varrho}{r} = \frac{s_1 - |BM|}{s-|BC|} = \frac{s_2 - |BM|}{s - |AB|} = \frac{s_1+s_2-2|BM|}{2s-|AB|-|BC|} = \frac{s-|BM|}{|AC|} \quad (2)$$

(1) i (2) zajedno daju:

$$(s-|BM|)(s+|BM|) = s \cdot |AC|$$

$$\Rightarrow |BM|^2 = s(s-|AC|) = s\cdot|BP| = s \cdot r \operatorname{ctg}\frac{\beta}{2} = P \cdot \operatorname{ctg}\frac{\beta}{2}$$

Zadaci za zadaću

DZ1: (kosinusov teorem za četverokut) Ako su a, b, c, d uzastopne stranice četverokuta, e i f njegove dijagonale, $\alpha = \sphericalangle(a,d)$, $\gamma = \sphericalangle(b,c)$, onda vrijedi

$$e^2 f^2 = a^2c^2 + b^2d^2 - 2abcd\cos(\alpha+\gamma)$$

DZ2: Neka je $ABCD$ konveksan četverokut. Ako je g najdulja, a h najkraća od udaljenosti $|AB|, |AC|, |AD|, |BC|, |BD|, |CD|$, dokaži da je tada $g \geq h\sqrt{2}$.

DZ3: Neka je L presjek simetrale kuta pri vrhu A u trokutu ABC s opisanom kružnicom. Dokaži da je

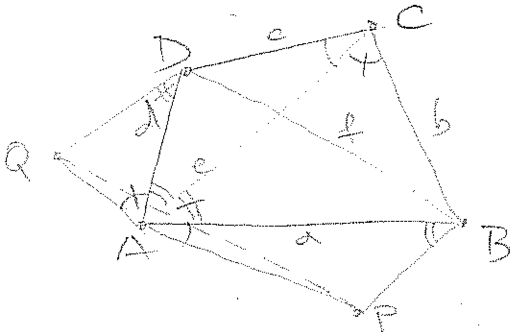
$$|AL| = \frac{b+c}{2\cos\frac{\alpha}{2}},$$

gdje je $\alpha = \sphericalangle BAC$, $b = |AC|$, $c = |AB|$.

DZ4: Neka su M, K, L točke koje se nalaze redom na stranicama $\overline{AB}, \overline{BC}, \overline{CA}$ trokuta ABC, pri čemu nijedna od točaka M, K, L nije vrh trokuta ABC. Dokazati da površina barem jednog od trokuta AML, BKM, CLK nije veća od $\frac{1}{4}$ površine trokuta ABC.

Rješenja zadataka za zadaću (TRIGONOMETRIJA)

DZ1:



Nad stranicom $\overline{AB}$ prema van konstruiramo $\Delta ABP \sim \Delta CAD$, tako da je $\sphericalangle PAB = \sphericalangle DCA$ i $\sphericalangle PBA = \sphericalangle DAC$ i nad $\overline{AD}$ prema van konstruiramo trokut $\Delta ADQ \sim \Delta CAB$, tako da je $\sphericalangle DAQ = \sphericalangle BCA$ i $\sphericalangle ADQ = \sphericalangle CAB$. Iz tih sličnosti slijedi da je

$$|AP| = \frac{ac}{e}, \quad |AQ| = \frac{bd}{e}, \quad |PB| = |DQ| = \frac{ad}{e}.$$

Nadalje je $\sphericalangle PBD + \sphericalangle QDB = \sphericalangle CAD + \sphericalangle ABD + \sphericalangle BDA + \sphericalangle CAB = 180^\circ$ pa je četverokut PBDQ paralelogram, a to povlači $|PQ| = |BD| = f$.
Dalje je $\sphericalangle PAQ = \alpha + \gamma$. Primjena kosinusovog poučka na ΔPAQ daje

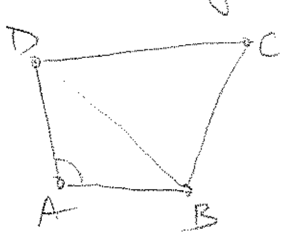
$$|PQ|^2 = |AP|^2 + |AQ|^2 - 2|AP| \cdot |AQ| \cos \sphericalangle PAQ.$$

Uvrštavanje navedenih vrijednosti daje

$$f^2 = \left(\frac{ac}{e}\right)^2 + \left(\frac{bd}{e}\right)^2 - 2 \cdot \frac{ac}{e} \cdot \frac{bd}{e} \cdot \cos(\alpha + \gamma)$$

pa odavde slijedi tvrdnja.

DZ2: Od svih kuteva četverokuta ABCD jedan je sigurno veći ili jednak od 90°. Neka je to npr. $\sphericalangle BAD$.



Tada iz kosinusovog poučka na ΔABD imamo:

$$|BD|^2 = |AB|^2 + |AD|^2 - 2|AB| \cdot |AD| \cos \sphericalangle BAD.$$

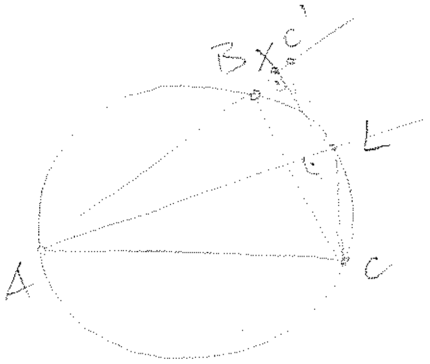
Odatle dobivamo:

$$|BD|^2 = |AB|^2 + |AD|^2 - 2|AB| \cdot |AD| \cdot \cos \sphericalangle BAD \geq |AB|^2 + |AD|^2 - 2|AB| \cdot |AD| \cdot \cos 90^\circ$$
$$= |AB|^2 + |AD|^2$$

Kako je $g \geq |AB|, |AD|, |BD| \geq h$, slijedi

$$g^2 \geq |BD|^2 \geq |AB|^2 + |AD|^2 \geq h^2 + h^2 = 2h^2$$
$$\Rightarrow g \geq h\sqrt{2}$$

DZ3:



Kako su u kružnici nad jednakim kutovima tetive jednakih duljina, to je $|BL|=|LC|$.

Neka je C' osnosimetrična slika točke C obzirom na pravac AL.
Očito će C' biti na pravcu AB jer je AL simetrala kuta i $|BL|=|LC'|$.
Neka je X okomica iz točke L na AB.
Kako je $\triangle BLC'$ jednakokračan, to će X biti polovište od $\overline{BC'}$ pa slijedi

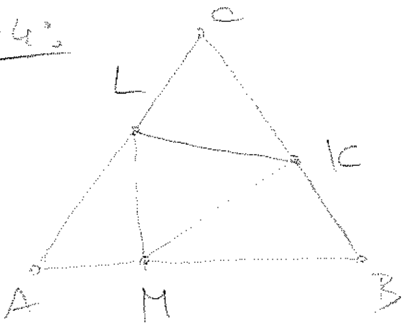
$$|AX|=|AB|+|BX|=|AB|+\frac{|BC'|}{2}=\frac{|AB|}{2}+\frac{|AB|+|BC'|}{2}=\frac{c}{2}+\frac{|AC'|}{2}=\frac{c}{2}+\frac{|AC|}{2}=\frac{b+c}{2}$$

Trokut AXL je pravokutan s kutom $\sphericalangle XAL=\frac{\alpha}{2}$ i zato je

$$|AX|=|AL|\cdot\cos\frac{\alpha}{2}$$

Konačno slijedi $|AL|=\dfrac{b+c}{2\cos\frac{\alpha}{2}}$.

DZ4:



Neka je $\frac{|AM|}{|AB|}=x$, $\frac{|BK|}{|BC|}=y$, $\frac{|CL|}{|CA|}=z$

$0<x,y,z<1$

Vrijedi $P_{AML}\cdot P_{BKM}\cdot P_{CLK}=\frac{1}{2}|AB|x\cdot|AC|(1-z)\sin\alpha\cdot\frac{1}{2}|AB|(1-x)\cdot|BC|y\sin\beta\cdot$

$$\cdot\frac{1}{2}|BC|(1-y)\cdot|AC|z\sin\gamma=$$

$$=\frac{1}{8}x(1-x)y(1-y)\cdot z(1-z)\cdot|AB|^2|AC|^2|BC|^2\sin\alpha\sin\beta\sin\gamma$$

$$\leq\frac{1}{8}\cdot\left(\frac{1}{4}\right)^3|AB|^2|AC|^2|BC|^2\sin\alpha\sin\beta\sin\gamma=\left(\frac{P}{4}\right)^3$$

($x(1-x)$, $y(1-y)$, $z(1-z)\leq\frac{1}{4}$ su očite nejednakosti)

To znači da je $\dfrac{4P_{AML}}{P}\cdot\dfrac{4P_{BKM}}{P}\cdot\dfrac{4P_{CLK}}{P}\leq 1$

pa je barem jedan od tih faktora manji ili jednak 1, a to se i tražilo u zadatku.